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Existence Theory of the Linear Equations Appearing in the Chapman–Enskog Solutions to Two Kinetic Equations for Liquids

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The linear operators appearing in the Chapman–Enskog solutions to Kirkwood's Fokker–Planck kinetic equation and to Rice and Allnatt's kinetic equation are studied in this article. Existence proofs are given for the linearized Chapman–Enskog equations involving either the Fokker–Planck or the Rice–Allnatt operators. It is shown that the Fokker–Planck and Rice–Allnatt operators, defined in the domain appropriate to kinetic theory, are essentially self-adjoint. It is also shown that the spectrum of either of these operators coincides with the spectrum of the self-adjoint extension of the corresponding operator.

KEY WORDS: Kinetic theory; nonequilibrium statistical mechanics; Fokker–Planck and Rice–Allnatt equations; existence theory for Chapman–Enskog solutions to kinetic equations.

1. INTRODUCTION

Several years ago, Kirkwood[1] developed a kinetic theory of simple fluids in which he introduced a second-order differential operator—sometimes called a Fokker–Planck collision operator—to account for the dissipative effect of intermolecular collisions on the rate of change of the singlet distribution function. In arriving at the Fokker–Planck operator, Kirkwood had to argue that collisions between molecules involved only small exchanges of momentum between molecules. Thus, he neglected in some sense the strongly repulsive collisions that molecules undergo when their intermolecular distances are such that the strongly repulsive forces are operative.

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Rice and Allnatt[2,3] introduced a model which accounted for "soft" collisions, involving small momentum exchanges, with a Fokker–Planck operator and for the "hard" close-range collisions with Enskog's hard-sphere collision operator, which is an integral operator. These two models have given a good deal of insight into the nature of transport processes in dense fluids and liquids. The Rice–Allnatt theory has led to encouraging agreement between predicted and experimental transport coefficients of the inert-gas fluids.

Our purpose here is not to review these theories but rather to prove the existence theorem which has been quoted without proof in studies of the Fokker–Planck and Rice–Allnatt kinetic equations.[2,3] A specious existence proof was published by Wei and Davis.[4] The theorem arises in connection with the Chapman–Enskog scheme[3,5] for solving the kinetic equations. The scheme leads to linear equations of the form

$$Au = f \tag{1}$$

where A is a linear operator. It is for these equations that an existence theorem is required.

The corresponding problem has been studied for dilute gases, in which case A is the linearized Boltzmann collision operator. Hecke[6] and later Carleman[7] proved the existence theorem for the Boltzmann operator in the case of molecules interacting with a hard-sphere potential, and Grad[8] proved the theorem for molecules interacting with potentials of finite range. All three investigators proved the existence theorem by showing that the operator A can be transformed to the form $A = E + K$, where E is the identity operator and K is a completely continuous operator. Then, the Fredholm alternative theorem[9] can be applied directly.

The operators appearing in Kirkwood's Fokker–Planck theory and the Rice–Allnatt theory involve second-order differential operators, so that the existence proof presented in this paper requires a different approach from that used by Hecke, Carleman, and Grad.

For Kirkwood's Fokker–Planck collision model, the operator in Eq. (1) may be written in the form[10] [3]

$$A_{\mathrm{FP}}u = -[1/\omega(\xi_1)]\,\nabla_{\boldsymbol{\xi}_1} \cdot [\omega(\xi_1)\,\nabla_{\boldsymbol{\xi}_1} u] - K_0[u] \tag{2a}$$

where

$$K_0[u] = \int d\boldsymbol{\xi}_2\,\omega(\xi_2)\,\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 u(\boldsymbol{\xi}_2) + \tfrac{1}{3}\int d\boldsymbol{\xi}_2\,\omega(\xi_2)(\xi_1^2 - 3)(\xi_2^2 - 3)\,u(\boldsymbol{\xi}_2) \tag{2b}$$

where $\boldsymbol{\xi}_i$ is the reduced momentum defined by

$$\boldsymbol{\xi}_i = \left(\frac{m}{kT}\right)^{1/2}\left(\frac{\mathbf{p}_i}{m} - \mathbf{v}_0\right) \tag{3}$$

[3] The terms composing $K_0[u]$ are usually suppressed in the definition of A_{FP} by appealing to the "auxiliary conditions."

and $\omega(\xi_i)$ is the Gaussian function

$$\omega(\xi_i) = [1/(2\pi)^{3/2}] \exp(-\xi_i^2/2) \tag{4}$$

$\mathbf{p}_i$ is the momentum of a reference particle, and $\mathbf{v}_0$ is the local hydrodynamic velocity. It will play no role in our discussion. Physically, the quantity u represents the deviation of the singlet distribution function from the Maxwellian distribution given by Eq. (4). The physical interpretation of u requires that the $u(\boldsymbol{\xi}_1)$ belong to the Hilbert space $\mathscr{L}^2(R_3 ; \omega)$, i.e., the space of functions for which the Lebesgue integral

$$\int d\boldsymbol{\xi}\, \omega(\xi) \mid u(\boldsymbol{\xi})|^2 \tag{5}$$

exists. The inner product (u, v) is defined in $\mathscr{L}^2(R_3 ; \omega)$ as

$$(u, v) = \int d\boldsymbol{\xi}\, \omega(\boldsymbol{\xi})\, \bar{u}(\boldsymbol{\xi})\, v(\boldsymbol{\xi}) \tag{6}$$

In Eq. (6), $\bar{u}$ denotes the complex conjugate of u.

For the Rice–Allnatt model, the operator in Eq. (1) may be written in the form

$$A_{\mathrm{RA}}u = A_{\mathrm{FP}}u + aL(u) \tag{7}$$

where a is a positive constant whose value need not be given here, and L is Boltzmann's collision operator for hard spheres. Grad[8] has shown that $L(u)$ can be written in the form

$$L(u) = \nu(\xi_1)\, u(\boldsymbol{\xi}_1) - K[u] \tag{8}$$

where

$$\nu(\xi_1) = (2\pi)^{1/2} \left[\exp(-\xi_1^2/2) + \xi_1 \int_0^{\xi_1} \exp(-\xi_2^2/2)\, d\xi_2\right] \tag{9}$$

and

$$K[u] = K_2[u] - K_1[u] \tag{10}$$

with

$$K_1[u] = \frac{1}{2(2\pi)^{1/2}\,[\omega(\xi_1)]^{1/2}} \int d\boldsymbol{\xi}_2 \mid \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \mid \left[\exp - \frac{1}{4}(\xi_1^2 + \xi_2^2)\right] [\omega(\xi_2)]^{1/2}\, u(\boldsymbol{\xi}_2) \tag{11}$$

$$K_2[u] = \frac{2}{(2\pi)^{1/2}\,[\omega(\xi_1)]^{1/2}} \int d\boldsymbol{\xi}_2 \frac{1}{\mid \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \mid} \left\{\exp\left[-\frac{1}{8} \mid \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \mid^2 - \frac{1}{8}\frac{(\xi_1^2 - \xi_2^2)^2}{\mid \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \mid^2}\right]\right\} \times [\omega(\xi_2)]^{1/2}\, u(\boldsymbol{\xi}_2) \tag{12}$$

where $d\boldsymbol{\xi}_2$ denotes a volume element in reduced momentum space.

The domains of both A_{FP} and A_{RA} are defined by the set of functions

$$\mathscr{D}_A = \{u(\boldsymbol{\xi}) \mid u(\boldsymbol{\xi}) \in C^2(R_3) \cap \mathscr{L}^2(R_3 ; \omega), \quad Au \in \mathscr{L}^2(R_3 ; \omega)\} \tag{13}$$

where $C^n(R_3)$ denotes the space of functions with continuous nth derivatives. The domain $\mathscr{D}_A$ is dense in the space $\mathscr{L}^2(R_3 ; \omega)$, which itself forms a Hilbert space $\mathscr{H}$ with the inner product given in Eq. (6).

In Section 2, we state the existence theorems whose proofs are presented in later sections. In Section 3, the essential self-adjointness and positive-semidefiniteness of A_{FP} and A_{RA} in $\mathscr{D}_A$ are established. In Section 4, an existence theorem is given for the self-adjoint extensions of A_{FP} and A_{RA}, and it is shown that this theorem implies the third existence theorem of Section 2. In Section 5, a Weyl lemma is proved for A_{FP} and A_{RA} in $\mathscr{D}_A$, and this lemma is used to prove the first existence theorem of Section 2. Finally, in Section 6, we investigate the spectra of A_{FP} and A_{RA} in $\mathscr{D}_A$ and prove the second existence theorem of Section 2.

2. THE EXISTENCE THEOREMS

As far as kinetic theory is concerned, the main results of this paper are three existence theorems. Let us state these theorems in this section and present the proofs of them in the sections to follow.

Theorem 1. For either $A = A_{\mathrm{FP}}$ or A_{RA}, Equation (1) has a solution $u \in \mathscr{D}_A$ for $f(\xi) \in C^1(R_3) \cap \mathscr{L}^2(R_3\,;\,\omega)$ if and only if

$$(f, \psi) = 0 \tag{14}$$

where ψ is any solution to the equation

$$A\psi = 0 \tag{15}$$

For both A_{FP} and A_{RA}, Eq. (15) has only five solutions. These are 1, ξ_1, and ξ_1^2 in each case.

The solution in the sense of Theorem 1 is a classical solution of Eq. (1). There are, however, two other types of solutions to Eq. (1) that are sometimes acceptable, or at least useful. First, we say that Eq. (1) admits a solution u in the *strong* $\mathscr{L}^2(R_3\,;\,\omega)$ *sense* when there exists a sequence $\{\varphi_n\}$ in $\mathscr{D}_A$ such that

$$\lim_{n\to\infty} \| A\varphi_n - f \| = 0 \tag{16}$$

and

$$\lim_{n\to\infty} \| \varphi_n - u \| = 0 \tag{17}$$

where $\| u \|$ $[\equiv (u, u)^{1/2}]$ denotes the norm of the function $u(\xi_1)$ in the space $\mathscr{L}^2(R_3\,;\,\omega)$. Second, we say that Eq. (1) has a solution u in the *weak* $\mathscr{L}^2(R_3\,;\,\omega)$ *sense* if the equation

$$(A\phi, u) = (\phi, f) \tag{18}$$

holds for every $\phi \in \mathscr{C}^\circ(R_3)$, where

$$\mathscr{C}^\circ(R_3) = \left\{ \phi(\xi_1) \,\middle|\, \begin{array}{l} \phi \in C^\infty(R_3), \quad \phi \equiv 0 \text{ outside a compact subset} \\ \text{of } R_3 \text{ depending on } \phi \text{ and contained in } R_3 \end{array} \right\} \tag{19}$$

The inner product in Eq. (18) is in the space $\mathscr{L}^2(R_3\,;\,\omega)$.

We can now state the other two existence theorems.

Theorem 2. For either $A = A_{\text{FP}}$ or A_{RA}, Eq. (1) has a solution u in the *strong* $\mathscr{L}^2(R_3\,;\,\omega)$ *sense* for $f(\xi_1) \in \mathscr{L}^2(R_3\,;\,\omega)$ if and only if

$$(f, \psi) = 0 \tag{20}$$

where ψ is any solution to Eq. (15).

Theorem 3. For either $A = A_{\text{FP}}$ or A_{RA}, Eq. (1) has a solution u in the *weak* $\mathscr{L}^2(R_3\,;\,\omega)$ *sense* for $f(\xi_1) \in \mathscr{L}^2(R_3\,;\,\omega)$ if and only if

$$(f, \psi) = 0 \tag{21}$$

where ψ is any solution to Eq. (15).

The functions f that occur in the Chapman–Enskog solution of the Fokker–Planck and Rice–Allnatt equations are at least once differentiable. Therefore, Theorem 1 appears to be sufficiently strong for the purposes of transport theory.

3. SOME PROPERTIES OF A_{FP} AND A_{RA} IN $\mathscr{D}_A$

Two properties of A_{FP} and A_{RA} in $\mathscr{D}_A$ are especially important in the proofs we shall give for Theorems 1–3. These are symmetry and essential self-adjointness. Consider an operator A defined in some domain of a Hilbert space $\mathscr{H}$. We say the operator A in $\mathscr{D}_A$ is *symmetric* if (1) $\mathscr{D}_A$ is dense in $\mathscr{H}$ and (2)

$$(u, Av) = (Au, v) \qquad \text{for all} \quad u, v \in \mathscr{D}_A \tag{22}$$

In many papers on kinetic theory, the symmetry condition (22) is erroneously called the condition of self-adjointness. Symmetry alone, however, does not imply self-adjointness.[11,12]

There are several equivalent definitions of the adjoint of an operator, but the definition which provides the clearest distinction between symmetric and self-adjoint operators is the following[12]: Let A in $\mathscr{D}_A$ be an operator and let $\mathscr{D}_A$ be dense in $\mathscr{H}$. Consider the elements $v \in \mathscr{H}$ and $v^* \in \mathscr{H}$ for which

$$(Au, v) = (u, v^*) \tag{23}$$

for all $u \in \mathscr{D}_A$. v^* is determined uniquely by v. The mapping of all v's into corresponding v^*'s defines the adjoint operator A^* of the operator A, and the functions v for which (23) holds define the domain $\mathscr{D}_{A^*}$ of A^*. Thus,

$$A^*v = v^* \qquad \text{for all} \quad v \in \mathscr{D}_{A^*} \tag{24}$$

and Eq. (23) reads

$$(Au, v) = (u, A^*v) \qquad \text{for all} \quad u \in \mathscr{D}_A \qquad \text{and for all} \quad v \in \mathscr{D}_{A^*} \tag{25}$$

The operator A in $\mathscr{D}_A$ is *defined to be self-adjoint* if A in $\mathscr{D}_A$ is equal to A^* in $\mathscr{D}_{A^*}$, that is, if the formal expressions for the operators A and A^* are identical and the domains $\mathscr{D}_A$ and $\mathscr{D}_{A^*}$ are identical. If A in $\mathscr{D}_A$ is symmetric, it is clear that $A = A^*$ in $\mathscr{D}_A$. However, the domain $\mathscr{D}_{A^*}$ may be larger than the domain $\mathscr{D}_A$, and, in fact, the form of A^* may be different from A for $v \in \mathscr{D}_{A^*}$ and $v \notin \mathscr{D}_A$. Thus, if we only know that A in $\mathscr{D}_A$ is symmetric, we can only say with certainty that $\mathscr{D}_A \subset \mathscr{D}_{A^*}$ and that $Au = A^*u$ for all $u \in \mathscr{D}_A$.

An *equivalent definition of self-adjointness*[12] is that (1) A in $\mathscr{D}_A$ is symmetric and (2) $(A + iE)\,\mathscr{D}_A = \mathscr{H}$ and $(A - iE)\,\mathscr{D}_A = \mathscr{H}$, i.e., the range of the operators $(A \pm iE)$ is the entire Hilbert space.

For symmetric differential operators, one often cannot establish self-adjointness but can establish the related and useful property of essential self-adjointness. We say an operator A in $\mathscr{D}_A$ is *essentially self-adjoint* if (1) A in $\mathscr{D}_A$ is symmetric and (2) $(A + iE)\,\mathscr{D}_A$ and $(A - iE)\,\mathscr{D}_A$ are dense in $\mathscr{H}$.

We shall prove in what follows that A_{FP} and A_{RA} are essentially self-adjoint in the $\mathscr{D}_A$ given by Eq. (13). *The property of essential self-adjointness of particular importance to us in this article is that the closure of A in $\mathscr{D}_A$ is self-adjoint.* The closure of A in $\mathscr{D}_A$, denoted herein as $\bar{A}$ in $\bar{\mathscr{D}}_A$, is defined as follows: $\bar{\mathscr{D}}_A$ consists of all $u \in \mathscr{H}$ for which there exist sequences $u_1, u_2, \ldots \in \mathscr{D}_A$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} Au_n$ converges in $\mathscr{H}$. For each such u, $\bar{A}$ is defined as $\bar{A}u = \lim_{n\to\infty} Au_n$. Clearly, $\mathscr{D}_A \subseteq \bar{\mathscr{D}}_A$ and $\bar{A}u = Au$ for all $u \in \mathscr{D}_A$. It is the closure $\bar{A}$ in $\bar{\mathscr{D}}_A$ that is involved in $\mathscr{L}^2$ solutions of Eq. (1).

Let us now turn to the study of A_{FP} and A_{RA} in $\mathscr{D}_A$ of Eq. (13). Here, the Hilbert space is $\mathscr{L}^2(R_3\,;\,\omega)$.

Theorem 4. The operators K_0, K_1, K_2 of Eqs. (2b), (11), and (12) are completely continuous self-adjoint operators in $\mathscr{L}^2(R_3\,;\,\omega)$.

Proof. For K_1 and K_2, the proof of the theorem has been given by Hecke,[6] Carleman,[7] and Grad[8] independently. It can be noted by inspection that K_0 is symmetric, and that it is an integral operator with a degenerate kernel composed of functions in $\mathscr{L}^2(R_3\,;\,\omega)$ and is, therefore, completely continuous.[9]

Theorem 5. The operators A_{FP} and A_{RA} in $\mathscr{D}_A$ are symmetric.

Proof. Both operators can be written in the form

$$A = S + K \tag{26}$$

where K is a completely continuous symmetric operator and S is a second-order differential operator of the form

$$Su = -[1/\omega(\xi_1)]\,\nabla_{\xi_1} \cdot [\omega(\xi_1)\,\nabla_{\xi_1} u] + a(\xi_1)\,u(\xi_1) \tag{27}$$

$$\begin{aligned} a(\xi_1) &= 0 && \text{in the case of } A_{\mathrm{FP}} \\ &= a\nu(\xi_1) && \text{in the case of } A_{\mathrm{RA}} \end{aligned} \tag{28}$$

Define the domain

$$\mathscr{D}_S = \{u(\xi_1) \mid u(\xi_1) \in C^2(R_3) \cap \mathscr{L}^2(R_3\,;\,\omega), \quad Su \in \mathscr{L}^2(R_3\,;\,\omega)\} \tag{29}$$

Since K is completely continuous and, consequently, $\| K[u] \| < \infty$ for any $u \in \mathscr{L}^2(R_3\,;\,\omega)$, the domain $\mathscr{D}_S$ is identical to the domain $\mathscr{D}_A$. Hellwig[12] has proved that differential operators of the form given by Eq. (27) are symmetric in $\mathscr{D}_S$ if $\omega(\xi_1) \in C^1(R_3)$ and $a(\xi_1) \in C^0(R_3)$. In fact, $\omega(\xi_1)$ and $a(\xi_1) \in C^\infty(R_3)$, so that Hellwig's theorem establishes that S in $\mathscr{D}_A$ is symmetric. Since K is symmetric in $\mathscr{L}^2(R_3\,;\,\omega)$ according to Theorem 4, $A = S + K$ is symmetric in $\mathscr{D}_A$.

We can now establish that A_{FP} and A_{RA} are essentially self-adjoint. The proof for A_{FP} is made particularly simple by the fact that the eigenfunctions of A_{FP} are known and form a complete orthonormal set in $\mathscr{L}^2(R_3\,;\,\omega)$. In the case of A_{RA}, as in Theorem 5, our work has already been done in Hellwig's studies[12] of a class of differential operators of which S is a special case.

Theorem 6. A_{FP} in $\mathscr{D}_A$ possesses the complete orthonormal set of eigenfunctions (see, e.g., Ref. 13)

$$\varphi_{rlm}(\xi) = N_{rlm}(\xi/\sqrt{2})^l\, S^{(r)}_{l+(1/2)}(\xi^2/2)\, Y_{lm}(\theta, \phi) \tag{30}$$

where $r,\ l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$. The eigenfunctions φ_{000}, φ_{01-1}, φ_{010}, φ_{011}, and φ_{100} correspond to the eigenvalue $\lambda_{rlm} = 0$, while for $r + l > 1$, the eigenvalues are $\lambda_{rlm} = 2r + l$; here, (ξ, θ, ϕ) is the spherical coordinate representation of ξ, $Y_{lm}(\theta, \phi)$ the spherical harmonic corresponding to lm, and $S^{(r)}_{l+(1/2)}(\xi^2/2)$ a Sonine polynomical defined by the relation

$$S_\nu^{(r)}(x) = \sum_{j=0}^{r} \frac{(-1)^j(\nu + r)!\, x^j}{(\nu + j)!\,(r - j)!\,j!} \tag{31}$$

In (30), N_{rlm} is a normalization constant,

$$N_{rlm} = [\pi^{3/2} r!/\Gamma(r + l + \tfrac{3}{2})]^{1/2} \tag{32}$$

Proof. From the facts that $\{Y_{lm}(\theta, \phi)\}_{lm}$ form a complete orthonormal set in $\mathscr{L}^2(0 < \theta < \pi, 0 < \phi < 2\pi; \cos\theta)$ and $\{(\xi/\sqrt{2})^l\, S^{(r)}_{l+(1/2)}(\xi^2/2)\}_{r=0}^{\infty}$ form a complete orthogonal set in $\mathscr{L}^2(0 < \xi < \infty; \xi^2\omega)$, it follows that $\{\varphi_{rlm}\}$ form a complete orthonormal set in $\mathscr{L}^2(R_3\,;\,\omega)$. It is easy to prove that φ_{rlm} are the eigenfunctions of A_{FP}. The procedure is to consider the eigenvalue problem

$$-[1/\omega(\xi_1)]\, \nabla_{\xi_1} \cdot [\omega(\xi_1)\, \nabla_{\xi_1}\varphi] = \lambda\varphi \tag{33}$$

The operator K_0 in A_{FP} is ignored for the moment. Equation (33) can be shown, by separation of variables, to admit the eigenvalues $\lambda_{rlm} = 2r + l$ and eigenfunctions φ_{rlm} given in Eq. (30). Then, by noting that

$$\begin{aligned} K_0[\varphi_{rlm}] &= \lambda_{rlm}\varphi_{rlm} \qquad &&\text{for}\quad rlm = 000, 01{-}1, 010, 011, 100 \\ &= 0 &&\text{otherwise} \end{aligned}$$

we see that Theorem 6 follows.

Theorem 6 will aid us in proving the next theorem.

Theorem 7. A_{FP} in $\mathscr{D}_A$ is essentially self-adjoint.

Proof. The proof is valid for any operator whose eigenfunctions form a complete orthonormal set in the Hilbert space. Denote the set of eigenfunctions of A_{FP} by $\{\varphi_j\}$. According to Theorem 6, the set is complete. We shall use this fact to prove that $(A_{FP} \pm iE)\mathscr{D}_A$ are dense in $\mathscr{L}^2(R_3 ; \omega)$, and, consequently, A_{FP}, is essentially self-adjoint, since symmetry was established by Theorem 5. Consider any $v \in \mathscr{L}^2(R_3 ; \omega)$ and define

$$v_n = \sum_{j=1}^{n} [1/(\lambda_j + i)](\varphi_j , v)\, \varphi_j \tag{34}$$

Since A_{FP} is symmetric, the λ_j are real, $v_n \in \mathscr{D}_{A_{FP}+iE}$, and the sequence $v_1, v_2, ...$ converges in $\mathscr{L}^2(R_3 ; \omega)$. Form the sequence $(A_{FP} + iE)\, v_n$ and note that

$$(A_{FP} + iE)\, v_n = \sum_{j=1}^{n} (\varphi_j , v)\, \varphi_j \xrightarrow[n\to\infty]{} v \tag{35}$$

almost everywhere since the set $\{\varphi_j\}$ is complete. The same treatment of $(A - iE)$ will complete the proof of the theorem.

Theorem 8. A_{RA} in $\mathscr{D}_A$ is essentially self-adjoint.

Proof. The symmetry of A_{RA} was established by Theorem 6. The rest of the proof follows directly from two theorems given by Hellwig.[12] We noted in the proof of Theorem 5 that A_{RA} can be written as the sum of a second-order differential operator S and a completely continuous operator K. The form of S is given in Eq. (26). The operator S in $\mathscr{D}_A$ belongs to a larger class of second-order differential operators which Hellwig has shown to be essentially self-adjoint (see his Theorem 2, Ref. 12, p. 189). Thus, S is essentially self-adjoint. The second theorem given by Hellwig that we wish to use here is as follows: If S in $\mathscr{D}$ is essentially self-adjoint, if K in $\mathscr{D}$ is symmetric, and if, for all $u \in \mathscr{D}$, $\| Ku \| \leqslant \epsilon \| Su \| + \delta \| u \|$ for some constants δ and ϵ with $0 \leqslant \epsilon < 1$, then $S + K$ in $\mathscr{D}$ is essentially self-adjoint. Since in our case K is completely continuous, there exists some constant a such that $\| Ku \| < a \| u \|$ for all $u \in \mathscr{L}^2(R_3 ; \omega)$. Thus, by choosing $\epsilon = 0$ and $\delta = a$, it follows that $A_{RA} = S + K$ in $\mathscr{D}_A$ is essentially self-adjoint.

The proof given in Theorem 8 also establishes the essential self-adjointness of A_{FP}, since this operator also can be decomposed into a sum of a second-order differential operator of the form treated by Hellwig and a completely continuous operator.

Another property of A_{FP} and A_{RA} that will be useful to us is positive-semi-definiteness. Let us list this property as a theorem without proving it.

Theorem 9. A_{FP} and A_{RA} in $\mathscr{D}_A$ are bounded from below by zero, i.e., for all $u \in \mathscr{D}_A$,

$$(u, Au) \geqslant 0 \tag{36}$$

for $A = A_{FP}$ or A_{RA}.

Later, in proving Theorem 1, it will be convenient to divide the operators A_{FP} and A_{RA} into a positive-definite invertible operator and a completely continuous operator. To this end, we need the following.

Theorem 10. For all $u \in \mathscr{D}_A$,

$$(u, A_{FP}u + a\nu(\xi)u) \geqslant b \| u \|^2 \tag{37}$$

where $b > 0$, and

$$(u, A_{FP}u + K_{00}[u]) \geqslant \| u \|^2 \tag{38}$$

where

$$K_{00}[u] = \varphi_{000}(\varphi_{000}, u) + \sum_{m=-1}^{1} \varphi_{01m}(\varphi_{01m}, u) + \varphi_{100}(\varphi_{100}, u) \tag{39}$$

Equation (37) follows from the facts that $a > 0$, $\nu(\xi) \geqslant (2\pi)^{1/2}/e$, and $(u, A_{FP}) \geqslant 0$. The proof of Eq. (38) is easy and will be omitted.

4. THE EXISTENCE THEOREM FOR $\bar{A}_{FP}$ AND $\bar{A}_{RA}$ IN $\bar{\mathscr{D}}_A$

In seeking a solution to Eq. (1) in the $\mathscr{L}^2$ sense, we may consider the closure equation

$$\bar{A}w = f \tag{40}$$

where $\bar{A}$ in $\bar{\mathscr{D}}_A$ is the closure of A in $\mathscr{D}_A$. We established in the last section that A_{FP} and A_{RA} in $\mathscr{D}_A$ are essentially self-adjoint. Therefore, $\bar{A}_{FP}$ and $\bar{A}_{RA}$ in $\bar{\mathscr{D}}_A$ are self-adjoint. Both A_{FP} and A_{RA} can be written in the form

$$A = B + C \tag{41}$$

where B is essentially self-adjoint in $\mathscr{D}_A$, and such that $(u, Bu) \geqslant b \| u \|^2$, $b > 0$, and C is completely continuous. In the case of A_{FP}, we choose

$$B_{FP} = A_{FP} + K_{00}, \qquad C_{FP} = -K_{00} \tag{42}$$

In the case of A_{RA}, we choose

$$B_{RA} = A_{FP} + \nu(\xi)E, \qquad C_{RA} = -[K_2 - K_1] \tag{43}$$

According to Theorem 10, B_{FP} and B_{RA} are bounded above zero, i.e., $(u, Bu) \geqslant b \| u \|^2$, $b > 0$. And, since K_1, K_2, and K_{00}, given in Eqs. (11), (12), and (39), respectively, are completely continuous, the operators C_{FP} and C_{RA} are completely continuous. The following existence theorem is applicable to Eq. (40) for both A_{FP} and A_{RA}.

Theorem 11. If B and C in $\mathscr{D}$ obey the conditions

(i) B self-adjoint;

(ii) $(w, Bw) \geqslant b \| w \|^2$, $b > 0$, $w \in \mathscr{D}$;

(iii) C completely continuous;

then the equation

$$(B + C)\, w = f \tag{44}$$

has a solution for $f \in \mathscr{L}^2$ if and only if

$$(f, \psi) = 0 \tag{45}$$

where ψ is any solution to the equation

$$(B + C)^* \psi = (B + C^*)\, \psi = 0 \tag{46}$$

where $(B + C)^*$ is the adjoint of $(B + C)$. Here, $(B + C)^*$ can be replaced with $B + C^*$ since B is self-adjoint and C is a bounded operator.[14]

Proof. Because of condition (ii), B^{-1} exists, and since B is self-adjoint, the spectral theorem implies that the domain of B^{-1} is the entire Hilbert space.[14] Thus, Eq. (44) may be partially inverted to obtain

$$w + B^{-1}Cw = B^{-1}f \tag{47}$$

If Eq. (47) has a solution, it will also be the solution of Eq. (44). But B^{-1} is a bounded operator, in view of condition (ii), and C is a completely continuous operator. Therefore, $B^{-1}C$ is a completely continuous operator. Thus, Eq. (47) is in the form for which the Fredholm alternative theorem is valid.[9] The theorem states that Eq. (47) has a solution if and only if

$$0 = (B^{-1}f, z) = (f, B^{-1}z) \tag{48}$$

where z is any solution of the equation

$$0 = (E + B^{-1}C)^* z = (E + C^*B^{-1})\, z = (B + C^*)\, B^{-1}z \tag{49}$$

$(E + B^{-1}C)^*$ may be replaced by $E + C^*B^{-1}$ because $B^{-1}C$ and C are bounded.

The second equality in Eq. (48) follows since B^{-1} is self-adjoint and $\mathscr{D}_{B^{-1}}$ is the entire Hilbert space. The rightmost equality of Eq. (49) demonstrates that the solubility condition (48) is equivalent to the solubility condition (45). This completes the proof of the theorem.

Theorem 11 applied to $\bar{A}_{\mathrm{FP}}$ and $\bar{A}_{\mathrm{RA}}$ states that Eq. (40) will have a solution for $f \in \mathscr{L}^2(R_3\,;\,\omega)$ if and only if

$$(f, \psi) = 0 \tag{50}$$

where ψ is any solution to the equation

$$\bar{A}\psi = 0 \tag{51}$$

For the operators A_{FP} and A_{RA}, it will be shown in Section 6 that the eigenfunctions of $\bar{A}$ and A are the same. Therefore, the solutions of Eq. (51) are

$$1, \quad \xi, \quad \text{and} \quad \xi^2 \tag{52}$$

or, orthogonalized (in polar coordinates), the solutions are

$$\varphi_{000}\,, \quad \varphi_{01-1}\,, \quad \varphi_{010}\,, \quad \varphi_{011}\,, \quad \text{and} \quad \varphi_{100} \tag{53}$$

for both $\bar{A}_{\mathrm{FP}}$ and $\bar{A}_{\mathrm{RA}}$.

At this point, if we accept Eq. (53) as the solutions to Eq. (52), we can prove Theorem 3 as a corollary to Theorem 11. Theorem 11 implies that the equation

$$(\phi, \bar{A}w) = (\phi, f) \tag{54}$$

has a solution for all $\phi \in \mathscr{C}^\circ(R_3)$ if and only if Eq. (40) holds for any solution to Eq. (51). Equation (54) can be rewritten as

$$(\bar{A}\phi, w) = (A\phi, w) = (\phi, f) \tag{55}$$

where we have used the symmetry of $\bar{A}$ and have noted that $\bar{A}\phi = A\phi$ for all $\phi \in \mathscr{C}^\circ(R_3)$. Equation (55) implies, then, that a solution to Eq. (40) is a solution to Eq. (1) in the weak $\mathscr{L}^2$ sense. Thus, Theorem 3 is proved.

5. THE PROOF OF THEOREM 1

In the preceding section, we saw that the equation $\bar{A}w = f$ has a solution for any $f \in \mathscr{L}^2(R_3\,;\,\omega)$ orthogonal to the null space of $\bar{A}^* = \bar{A}$, i.e., for any f such that $(f, \psi) = 0$, where ψ is any solution to the equation $\bar{A}\psi = 0$. In this section, we shall show that if, in addition to the orthogonality condition on f, we require that $f \in C^1(R_3) \cap \mathscr{L}^2(R_3\,;\,\omega)$, then $w \in C^2(R_3) \cap \mathscr{L}^2(R_3\,;\,\omega)$, and, therefore, a solution to $\bar{A}w = f$ is a solution to our original equation, since $Au = \bar{A}u$ for any $u \in C^2(R_3) \cap \mathscr{L}^2(R_3\,;\,\omega)$. Then, by establishing that any solution ψ of the equation $\bar{A}\psi = 0$ is also a solution of the equation $A\psi = 0$, we shall complete the proof of Theorem 1.

To establish the desired differentiability of w and ψ, we shall need a theorem which is a generalization of a well-known lemma proved by Weyl in connection with second-order differential operators. Before proving this theorem, which appears below as Theorem 15, let us examine further some properties of the operators composing A_{FP} and A_{RA} in $\mathscr{D}_A$.

Up to this point, we have been considering the solution of Eq. (1) in the space $\mathscr{L}^2(R_3\,;\,\omega)$. It is expedient now to introduce the isometric transformation $T = \omega^{1/2}(\xi)$ which transforms our problem to a problem in the space of simple Lebesgue measure $\mathscr{L}^2(R_3)$. According to the transformation T, the functions $u \in \mathscr{L}^2(R_3\,;\,\omega)$ are transformed to functions $\hat{u} \in \mathscr{L}^2(R_3)$ by the mapping

$$\hat{u}(\xi) = Tu = \omega^{1/2}(\xi)\, u(\xi) \tag{56}$$

and the operators A in $\mathscr{D}_A$ are transformed to the operators $\hat{A}$ in $\mathscr{D}_{\hat{A}}$, where

$$\hat{A} = TAT^{-1} \tag{57}$$

and

$$\mathscr{D}_{\hat{A}} = \{\hat{u} \mid \hat{u} \in C^2(R_3) \cap \mathscr{L}^2(R_3), \quad \hat{A}\hat{u} \in \mathscr{L}^2(R_3)\} \tag{58}$$

Equation (1) is transformed to the equation

$$\hat{A}\hat{u} = \hat{f} \tag{59}$$

The explicit forms of $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$ are

$$\hat{A}_{\mathrm{FP}}\hat{u} = -\Delta_{\xi_1}\hat{u} + (\tfrac{1}{4}\xi_1{}^2 - \tfrac{3}{2})\,\hat{u} - \hat{K}_0[\hat{u}] \tag{60}$$

with

$$\hat{K}_0[\hat{u}] = \omega^{1/2}(\xi_1)\int_{R_3} d\boldsymbol{\xi}_2\,\omega^{1/2}(\xi_2)\,\boldsymbol{\xi}_1\cdot\boldsymbol{\xi}_2\hat{u}(\boldsymbol{\xi}_2)$$
$$+ \tfrac{1}{3}\omega^{1/2}(\xi_1)\int_{R_3} d\boldsymbol{\xi}_2\,\omega^{1/2}(\xi_2)(\xi_1{}^2 - 3)(\xi_2{}^2 - 3)\,\hat{u}(\boldsymbol{\xi}_2), \tag{61}$$

and

$$\hat{A}_{\mathrm{RA}}\hat{u} = -\Delta_{\xi_1}\hat{u} + [av(\xi_1) + \tfrac{1}{4}\xi_1{}^2 - \tfrac{3}{2}]\,\hat{u} - \hat{K}_0[\hat{u}] - a\hat{K}_2[\hat{u}] + a\hat{K}_1[\hat{u}] \tag{62}$$

with

$$\hat{K}_1[\hat{u}] = \frac{1}{2(2\pi)^{1/2}}\int_{R_3} d\boldsymbol{\xi}_2\,\xi_{12}\left[\exp -\frac{1}{4}(\xi_1{}^2 + \xi_2{}^2)\right]\hat{u}(\boldsymbol{\xi}_2) \tag{63}$$

$$\hat{K}_2[\hat{u}] = \frac{2}{(2\pi)^{1/2}}\int_{R_3} d\boldsymbol{\xi}_2\,\frac{1}{\xi_{12}}\left\{\exp\left[-\frac{1}{8}\xi_{12}^2 - \frac{1}{8}\frac{(\xi_1{}^2 - \xi_2{}^2)^2}{\xi_{12}^2}\right]\right\}\hat{u}(\boldsymbol{\xi}_2) \tag{64}$$

We introduced the notations

$$\xi_{12} \equiv |\,\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\,| \tag{65}$$

$$\Delta_{\xi_1} \equiv \nabla_{\xi_1}\cdot\nabla_{\xi_1} \tag{66}$$

Theorems 3–11, proved for A_{FP} and A_{RA} in $\mathscr{D}_A$, remain true for $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$ in $\mathscr{D}_{\hat{A}}$. Moreover, if we can establish Theorems 1 and 2 for $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$ in $\mathscr{D}_{\hat{A}}$, then they will be true for A_{FP} and A_{RA} in $\mathscr{D}_A$.

In proving Theorem 15, we shall use certain smoothness properties of the functions generated by $\hat{K}_0[\hat{u}]$, $\hat{K}_1[\hat{u}]$, and $\hat{K}_2[\hat{u}]$. Let us establish these properties now. A useful result is the well-known smoothness theorem[15] involving the singular solution of the equation $\Delta_{\mathbf{x}}v = 0$. We shall present the theorem here for functions defined in an n-dimensional vector space for $n > 2$. Thus, $\mathbf{x}$ denotes a vector in R_n, $d\mathbf{x}$ a volume element in R_n and $\Delta\mathbf{x}$ the n-dimensional Laplacian. If D denotes a normal domain in R_n and $\bar{D}$ its closure, the theorem reads as follows.

Theorem 12. (1) If $f(x) \in C^0(D)$ and is bounded in D, then the function $u(\mathbf{x})$, defined by the equation

$$u(\mathbf{x}) = -\int_D s(\mathbf{x}, \mathbf{y})\,f(\mathbf{y})\,d\mathbf{y} \tag{67}$$

where

$$s(\mathbf{x}, \mathbf{y}) = [1/(n-2)\,\omega_n]\,|\,\mathbf{x} - \mathbf{y}\,|^{2-n}, \qquad n > 2 \tag{68}$$

(ω_n the area of a unit sphere in R_n) belongs to $C^1(\bar{D})$, and, in particular,

$$\partial u/\partial x_i = -\int_D [\partial s(\mathbf{x}, \mathbf{y})/\partial x_i] f(\mathbf{y})\, d\mathbf{y} \tag{69}$$

where x_i is the ith Cartesian coordinate of $\mathbf{x}$. (2) If $f(x) \in C^1(\bar{D})$, then the function $u(\mathbf{x})$ defined by Eq. (67) belongs to $C^1(\bar{D})$, and to $C^2(D)$, and satisfies the equation

$$\Delta_{\mathbf{x}} u = f(\mathbf{x}) \qquad \text{for} \quad x \in D \tag{70}$$

Theorem 13. The functions $\hat{K}_0[\hat{u}]$, $\hat{K}_1[\hat{u}]$, and $\hat{K}_2[\hat{u}]$ are bounded on R_3 for any function $\hat{u} \in \mathscr{L}^2(R_3)$.

Proof. The validity of the theorem is obvious for $\hat{K}_0[\hat{u}]$. By use of the Schwarz inequality, we see

$$\hat{K}_1[\hat{u}] \leqslant [\| \hat{u} \|/2(2\pi)^{1/2}] \left[\int d\xi_2\, \xi_{12}^2 \exp - \tfrac{1}{2}(\xi_1{}^2 + \xi_2{}^2)\right]^{1/2} < +\infty \qquad \text{for all} \quad \xi_1 \in R_3 \tag{71}$$

$$\hat{K}_2[\hat{u}] \leqslant [2 \| \hat{u} \|/(2\pi)^{1/2}] \left[\int d\xi_2\, (1/\xi_{12}^2) \exp - \tfrac{1}{4}\xi_{12}^2\right]^{1/2} < +\infty \qquad \text{for all} \quad \xi_1 \in R_3 \tag{72}$$

Thus, the theorem is true.

Theorem 14. If $\hat{u}(\xi) \in C^0(R_3) \cap \mathscr{L}^2(R_3)$, then the functions $\hat{K}_0[\hat{u}]$, $\hat{K}_1[\hat{u}]$, and $\hat{K}_2[\hat{u}] \in C^1(R_3)$.

Proof. Again, the truth of the theorem is obvious for $\hat{K}_0[\hat{u}]$. In fact, for any $\hat{u} \in \mathscr{L}^2(R_3)$, $\hat{K}_0[\hat{u}] \in C^\infty(R_3)$. The proof is straightforward for $\hat{K}_1[\hat{u}]$. Thus, we shall treat only $\hat{K}_2[\hat{u}]$. Let us denote by $\{D_n\}$ a sequence of concentric balls having radii $\{\rho_n\}$ and centered about an arbitrary element in R_3. We choose the balls such that

$$D_0 \subset D_1 \subset \cdots$$

and

$$\rho_0 \leqslant \rho_1 < \rho_2 \cdots, \qquad \rho_n \to \infty \quad \text{as} \quad n \to \infty$$

so that

$$\lim_{n\to\infty} D_n = R_3$$

Define the sequence $\{v_n(\xi)\}$ by

$$v_n(\xi_1) = \frac{2}{(2\pi)^{1/2}} \int_{D_n} d\xi_2 \frac{1}{\xi_{12}} \left\{\exp\left[-\frac{1}{8}\xi_{12}^2 - \frac{1}{8}\frac{(\xi_1{}^2 - \xi_2{}^2)^2}{\xi_{12}^2}\right]\right\} \hat{u}(\xi_2) \tag{73}$$

This is a converging sequence. In fact,

$$v(\xi_1) = \lim_{n\to\infty} v_n(\xi_1) = \hat{K}_2[\hat{u}] \tag{74}$$

According to Theorem 12 (generalized slightly for the case at hand), $v_n(\boldsymbol{\xi}_1) \in C^1(\bar{D}_n)$, and, in particular, for the ith Cartesian component of $\boldsymbol{\xi}_1$,

$$\frac{\partial v_n(\boldsymbol{\xi}_1)}{\partial \xi_{1i}} = \frac{2}{(2\pi)^{1/2}} \int_{D_n} d\boldsymbol{\xi}_2 \Big[-\frac{\xi_{1i} - \xi_{2i}}{\xi_{12}^3} - \frac{1}{4} \frac{(\xi_{1i} - \xi_{2i})}{\xi_{12}} - \frac{1}{2} \xi_{1i} \frac{(\xi_1^2 - \xi_2^2)}{\xi_{12}^2}$$
$$+ \frac{1}{4} \frac{(\xi_{1i} - \xi_{2i})(\xi_1^2 - \xi_2^2)^2}{\xi_{12}^5} \Big] \Big\{ \exp\Big[-\frac{1}{8} \xi_{12}^2 - \frac{1}{8} \frac{(\xi_1^2 - \xi_2^2)^2}{\xi_{12}^2} \Big] \Big\} \hat{u}(\boldsymbol{\xi}_2) \quad (75)$$

Assume now that $\boldsymbol{\xi}_1 \in D_0$. Then, for $n > m > 0$, we obtain the inequality

$$\left| \frac{\partial v_n(\boldsymbol{\xi}_1)}{\partial \xi_{1i}} - \frac{\partial v_m(\boldsymbol{\xi}_1)}{\partial \xi_{1i}} \right| \leqslant \frac{c}{| \rho_0 - \rho_m |} \qquad \text{for all} \quad \boldsymbol{\xi}_1 \in D_0 \quad (76)$$

where $c > 0$ is a constant. This inequality is obtained by noting that the domain of the $\boldsymbol{\xi}_2$-integration of the difference between the derivatives of v_n and v_m is the concentric space between D_n and D_m and that the largest value $1/\xi_{12}$ can assume in this domain is $1/| \rho_0 - \rho_m |$ corresponding to the smallest value ξ_{12} can assume. This fact plus the Schwarz inequality yield Eq. (57). Since $\rho_m \to \infty$ as $m \to \infty$, Eq. (76) implies that the sequence $\{\partial v_n(\boldsymbol{\xi}_1)/\partial \xi_{1i}\}$ converges uniformly for $\boldsymbol{\xi}_1 \in D_0$. For such a sequence, a well-known theorem of calculus[16] tells us that the limit of the sequence exists and, moreover, that the limit and differentiation can be interchanged, i.e.,

$$\lim_{n\to\infty} \frac{\partial v_n(\boldsymbol{\xi}_1)}{\partial \xi_{1i}} = \frac{\partial v(\boldsymbol{\xi}_1)}{\partial \xi_{1i}} = \frac{\partial}{\partial \xi_{1i}} \{\hat{K}[\hat{u}]\} \qquad \text{for all} \quad \boldsymbol{\xi}_1 \in D_0 \quad (77)$$

But since the origin of the ball D_0 is arbitrary, we can conclude that $\hat{K}[\hat{u}] \in C^1(R_3)$.

Now we come to the main theorem in this section. Since no extra work is involved, we shall give the theorem for functions defined on an n-dimensional vector space. Thus, for the purposes of this theorem, $\boldsymbol{\xi} \in R_n$ and $d\boldsymbol{\xi}$ is a volume element in R_n. Let us denote by G any open, simply connected set in R_n, in particular $G = R_n$ is admissible. Define

$$\mathscr{C}^\circ(G) = \left\{ \phi(\boldsymbol{\xi}) \,\middle|\, \begin{array}{l} \phi(\boldsymbol{\xi}) \in C^\infty(G), \quad \phi \equiv 0 \\ \text{outside a compact subset of } R_n \text{, depending} \\ \text{on } \phi \text{ and contained in } G \end{array} \right\} \quad (78)$$

A function $w(\boldsymbol{\xi})$ is said to be locally integrable in G if $\int_D w(\boldsymbol{\xi})\, d\boldsymbol{\xi} < +\infty$ for every compact subset D of R_n contained in G. In particular, any function in $\mathscr{L}^2(R_n)$ is locally integrable. Let us now state the theorem.

Theorem 15. Let $\eta(\boldsymbol{\xi}) \in C^1(G)$ and let $w(\boldsymbol{\xi})$ be a locally integrable function in G. Let K be a completely continuous operator for which $K[u]$ is a bounded function for any $u \in \mathscr{L}^2(G)$, $K[u] \in C^1(G)$ for any $u \in C^0(G) \cap \mathscr{L}^2(G)$, and $K[u] \in C^2(G)$ for any $u \in C^1(G) \cap \mathscr{L}^2(G)$. Let $a(\boldsymbol{\xi})$ be a function belonging to $C^1(G)$. If the relation

$$\int_G \{-w(\boldsymbol{\xi})\, \Delta_{\boldsymbol{\xi}} \phi + a(\boldsymbol{\xi})\, w(\boldsymbol{\xi})\, \phi(\boldsymbol{\xi}) - \phi(\boldsymbol{\xi})\, K[w]\}\, d\boldsymbol{\xi} = \int_G \eta(\boldsymbol{\xi})\, w(\boldsymbol{\xi})\, d\boldsymbol{\xi} \quad (79)$$

for every $\phi \in \mathscr{C}^\circ(G)$, then $w(\xi)$ is equal almost everywhere to a function belonging to $C^2(G)$, i.e., in a Hilbert space we may assert $w(\xi) \in C^2(G)$. Here, Δ_ξ denotes the n-dimensional Laplacian operator.

Theorem 15 is a generalization of what is known as a Weyl lemma for elliptical differential equations. The lemma was first proved by Weyl[17] for the n-dimensional Laplacian operator. Later, similar theorems have been obtained for a variety of differential equations. In particular, Fichera[18] and Wienholtz[19] have presented elementary proofs of the lemma for elliptical differential operators. Their method has been outlined in detail by Hellwig[12] for the operator $-\Delta_{\mathbf{x}} + a(\mathbf{x})$, where $\Delta_{\mathbf{x}}$ is an n-dimensional Laplacian operator. The proof of Theorem 15 follows as an easy extension of the method, as outlined by Hellwig on pages 193–197 of Ref. 12.

Proof. To try to conserve space, we shall follow as closely as possible the development given by Hellwig on pages 193–197 of Ref. 12. Equations of that treatment will be referred to in the form H.Eq. $(\cdot)$.

Let D_1 be a ball centered around an arbitrary point in G and such that $\bar{D}_1 \subset G$. Let D_2 be a ball concentric to D_1 such that $\bar{D}_2 \subset D_1$. Let $\rho(\mathbf{x}) \in \mathscr{C}^\circ(D_1)$ and let $\rho(\mathbf{x}) \equiv 1$ in D_2. Define for $\mathbf{y} \in D_2$ the function

$$v(\mathbf{y}) = \int_{D_1} w(\mathbf{x})\, \Delta_{\mathbf{x}}[\rho(\mathbf{x})\, s(\mathbf{y}, \mathbf{x})]\, d\mathbf{x} + \int_{D_1} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x})\{\eta(\mathbf{x}) - a(\mathbf{x})\, w(\mathbf{x}) + K[w]\}\, d\mathbf{x} \tag{80}$$

The problem now is first to prove that $w(\mathbf{x})$ is equal to $v(\mathbf{x})$ almost everywhere and then to deduce from the properties of the right-hand side of Eq. (80) that $w(\mathbf{x})$ is equal almost everywhere to a function in C^2. By replacing the entity $[\eta - aw]$ with $\{\eta - aw + K[w]\}$ everywhere it appears in H.Eqs. (4)–(10), the arguments in Ref. 12 follow directly for the present case, and we arrive at the result

$$\int_{D_2} [v(\mathbf{y}) - w(\mathbf{y})]\, \phi(\mathbf{y})\, d\mathbf{y} = -\int_{D_1} \{-w(\mathbf{x})\, \Delta_{\mathbf{x}}\psi + a(\mathbf{x})\, w(\mathbf{x})\, \psi(\mathbf{x}) - \psi(\mathbf{x})\, K[w]\}\, d\mathbf{x} \tag{81}$$

where $\phi(\mathbf{y})$ is any function in $\mathscr{C}^\circ(G)$ and $\psi(\mathbf{x}) = \rho(\mathbf{x}) \int_{D_2} s(\mathbf{y}, \mathbf{x})\, \phi(\mathbf{y})\, d\mathbf{y} \in C^2(D_1)$, by Theorem 12, and vanishes identically in a neighborhood of the surface of D_1. Thus, $\psi(\mathbf{x})$ and its first and second derivatives can be approximated uniformly by functions $\phi \in \mathscr{C}^\circ(G)$. Consequently, by the hypothesis of the theorem, the right-hand side of Eq. (81) vanishes, implying that

$$\int [v(\mathbf{y}) - w(\mathbf{y})]\, \phi(\mathbf{y})\, d\mathbf{y} = 0 \qquad \text{for all} \quad \phi \in \mathscr{C}^\circ(D_2) \tag{82}$$

which, in turn, implies that $v(\mathbf{y}) = w(\mathbf{y})$ almost everywhere in D_2.

Thus,

$$w(\mathbf{y}) = \int_{D_1} w(\mathbf{x})\, \Delta_{\mathbf{x}}[\rho(\mathbf{x})\, s(\mathbf{y}, \mathbf{x})]\, d\mathbf{x} + \int_{D_1} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x}) \{\eta(\mathbf{x}) - a(\mathbf{x})\, w(\mathbf{x}) + K[w]\}\, d\mathbf{x} \tag{83}$$

almost everywhere in D_2. The integrand of the first term on the rhs of (83) is zero except in the domain $D_1 - D_2$ owing to facts that $\rho(\mathbf{x}) \equiv 1$ for $\mathbf{x} \in D_2$ and $\Delta_{\mathbf{x}} s(\mathbf{y}, \mathbf{x}) = 0$ for $\mathbf{x} \neq \mathbf{y}$. Thus, the first term is $\int_{D_1 - D_2} \cdot\, w(\mathbf{x})\, \Delta_{\mathbf{x}}[\rho(\mathbf{x})\, s(\mathbf{y}, \mathbf{x})]\, d\mathbf{x}$, which belongs to $C^{\infty}(D_2)$. The integral involving $\eta(\mathbf{x})$ belongs to $C^2(D_1)$ according to Theorem 12. Equation (83) can consequently be written in the form

$$w(\mathbf{y}) = -\int_{D_1} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x}) \{a(\mathbf{x})\, w(\mathbf{x}) - K[w]\}\, d\mathbf{x} + g_1(\mathbf{y}) \tag{84}$$

where $\bar{D}_3 \subset D_2$ and $g_1(\mathbf{y}) \in C^2(\bar{D}_3)$, almost everywhere in D_3.

From the hypothesis that $K[w]$ is a bounded function in R_n and $\rho(\mathbf{x})$ is bounded in D_1, we can conclude that

$$\left| \int_{D_1} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x})\, K[w]\, d\mathbf{x} \right| \leqslant c \qquad \text{for all} \quad y \in R_n \tag{85}$$

where c is a constant. Hence, the inequality

$$| w(\mathbf{y}) | \leqslant \int_{D_1} \frac{c_1\, | w(\mathbf{x}) |}{| \mathbf{y} - \mathbf{x} |^{n-2}}\, d\mathbf{x} + c_2 \tag{86}$$

holds almost everywhere in D_3, where c_1 and c_2 are constants. Equation (86) is identical to H.Eq. (14), and, therefore, H.Eqs. (15)–(22) imply that there exists a ball D_α, $\bar{D}_\alpha \subset D_1$, in which $w(\mathbf{x})$ is equal almost everywhere to a function $b(\mathbf{x})$ which is bounded in D_2, and is integrable since $w(\mathbf{x})$ is integrable. The boundedness of $w(\mathbf{x})$ in D_2 is established by iteration of the inequality in Eq. (86).

Consider the balls $D_{\alpha+1}$, $D_{\alpha+2}$ concentric to D_α and such that $\bar{D}_{\alpha+2} \subset D_{\alpha+1} \subset \bar{D}_{\alpha+1} \subset D_\alpha$. Be defining $\rho(\mathbf{x}) \in (D_\alpha)$ such that $\rho(\mathbf{x}) \equiv 1$ in $D_{\alpha+1}$, we obtain, analogously to Eq. (84), the expression

$$w(\mathbf{y}) = -\int_{D_\alpha} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x}) \{a(\mathbf{x})\, b(\mathbf{x}) - K[w]\}\, d\mathbf{x} + g_\alpha(\mathbf{y}) \tag{87}$$

almost everywhere in $D_{\alpha+2}$. Here, $g_\alpha(\mathbf{y}) \in C^2(\bar{D}_{\alpha+2})$. Since $\rho(\mathbf{x})\, a(\mathbf{x})\, b(\mathbf{x})$ and $\rho(\mathbf{x})\, K[w]$ are bounded in D_α, we see from Theorem 12 that $w(\mathbf{y})$ coincides almost everywhere in $D_{\alpha+2}$ with a function $c(\mathbf{x}) \in C^1(\bar{D}_{\alpha+2})$. But because the original ball was centered on an arbitrary point in G, we conclude that

$$w(\mathbf{x}) \in C^1(G) \tag{88}$$

almost everywhere in G.

Repeating the argument used to obtain Eq. (87), we obtain

$$w(\mathbf{y}) = -\int_{D_{\alpha+2}} s(\mathbf{y}, \mathbf{x})\, \rho(\mathbf{x})\{a(\mathbf{x})\, w(\mathbf{x}) - K[w]\}\, d\mathbf{x} + g_{\alpha+2}(\mathbf{y}) \tag{89}$$

where, almost everywhere in $D_{\alpha+4}$,

$$g_{\alpha+2}(\mathbf{y}) \in C^2(\bar{D}_{\alpha+4}), \qquad \bar{D}_{\alpha+4} \subset D_{\alpha+3} \subset \bar{D}_{\alpha+3} \subset D_{\alpha+2}, \qquad \text{and} \qquad \rho(\mathbf{x}) \in \mathscr{C}^\circ(D_{\alpha+3})$$

with $\rho(\mathbf{x}) \equiv 1$ in $D_{\alpha+4}$. According to Eq. (88) and the hypothesis of the theorem, $\rho(\mathbf{x})\{a(\mathbf{x})\, w(\mathbf{x}) - K[w]\} \in C^1(D_{\alpha+2})$. Application again of Theorem 12 leads to the result $w(\mathbf{x}) \in C^2(D_{\alpha+4})$ almost everywhere. Noting again that the sphere $D_\perp$ was centered around an arbitrary point in G, we conclude that $w(x) \in C^2(G)$ almost everywhere in G. This concludes the proof of the theorem.

We are now in position to prove Theorem 1. According to Theorem 11, for either $\bar{\hat{A}}_{\mathrm{FP}}$ or $\bar{\hat{A}}_{\mathrm{RA}}$ in $\mathscr{D}_{\hat{A}}$, the equation

$$\bar{\hat{A}}\hat{w} = \hat{f} \tag{90}$$

has a solution in $\mathscr{L}^2(R_3)$ for $\hat{f} \in \mathscr{L}^2(R_3)$ if and only if

$$(\hat{f}, \hat{\psi}) = 0 \tag{91}$$

where $\hat{\psi}$ is any solution of the equation

$$\bar{\hat{A}}\hat{\psi} = 0 \tag{92}$$

Multiplying Eq. (90) by $\hat{\phi} \in \mathscr{C}^\circ(R_3)$ and integrating, we obtain

$$(\hat{\phi}, \bar{\hat{A}}\hat{w}) = (\hat{\phi}, \hat{f}) \qquad \text{for any} \quad \hat{\phi} \in \mathscr{C}^\circ(R_3) \tag{93}$$

Since $\bar{\hat{A}}$ is self-adjoint and since $\bar{\hat{A}}\hat{\phi} = \hat{A}\hat{\phi}$ for any $\hat{\phi} \in \mathscr{C}^\circ(R_3)$, Eq. (93) can be written in the form

$$\begin{aligned} &\int_{R_3} d\xi_1 \{-\hat{w}(\xi_1)\, \Delta_{\xi_1}\hat{\phi} + a(\xi_1)\, \hat{w}(\xi_1)\, \hat{\phi}(\xi_1) - \hat{\phi}(\xi_1)\, \hat{K}[\hat{w}]\}\, d\xi_1 \\ &\qquad = \int_{R_3} d\xi_1\, \hat{\phi}(\xi_1) \hat{f}(\xi_1) \end{aligned} \tag{94}$$

for all $\hat{\phi} \in \mathscr{C}^\circ(R_3)$. Referring to the definitions of $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$, Eqs. (60)–(64), and to the conclusions of Theorems 13 and 14, we see that $a(\xi_1)$ and $\hat{K}$ satisfy the hypotheses of Theorem 15. Therefore, if $\hat{f}(\xi_1) \in C^1(R_3)$, it follows from Theorem 15 that

$$\hat{w}(\xi_1) \in C^2(R_3) \qquad \text{or} \qquad \hat{w}(\xi_1) \in \mathscr{D}_{\hat{A}}$$

and, therefore, $\bar{\hat{A}}\hat{w} = \hat{A}\hat{w} = \hat{f}$, establishing the existence of a solution to $Au = f$ of Theorem 1. Applying Theorem 15 for the case $\hat{f} \equiv 0$, we conclude that $\hat{\psi} \in C^2(R_3)$, where $\hat{\psi}$ is any solution to Eq. (92), so that

$$\hat{A}\hat{\psi} = 0$$

establishing Eq. (15). This completes the proof of Theorem 1.

6. SOME SPECTRAL PROPERTIES OF A_{FP} AND A_{RA}

Theorem 15 and a closely related theorem, which we shall give later, allow us to say a great deal about the spectrum of A_{FP} and A_{RA}. We have already seen that the eigenfunctions of $\bar{A}_{\mathrm{FP}}$ and $\bar{A}_{\mathrm{RA}}$ corresponding to the zero eigenvalue are also eigenfunctions of A_{FP} and A_{RA}. We shall give the even stronger result that every eigenfunction of $\bar{A}_{\mathrm{FP}}$ and $\bar{A}_{\mathrm{RA}}$, respectively, is an eigenfunction of A_{FP} and A_{RA}, respectively. That is the content of the following theorem.

Theorem 16. If A in $\mathscr{D}_A$ is an operator of the form given in the hypotheses of Theorem 15, then, for every eigenfunction $u(\mathbf{x}) \in \bar{\mathscr{D}}_A$,

$$u(\mathbf{x}) \in \mathscr{D}_A, \qquad \text{i.e.,} \quad \bar{A}u = Au \tag{95}$$

Proof. Assume $u \in \mathscr{L}^2(R_3)$ is the eigenfunction of $\bar{A}$ in $\bar{\mathscr{D}}_A$ corresponding to the eigenvalue λ. Then,

$$\bar{A}u - \lambda u = 0$$

or

$$(\phi, Au - \lambda u) = 0 \qquad \text{for all} \quad \phi \in \mathscr{C}^\circ(R_3) \tag{96}$$

If A satisfies the hypothesis of Theorem 15, so does $A - \lambda E$, and, therefore, Eq. (96) is the special case for which $f \equiv 0$ in Theorem 15. Hence, $u(x) \in C^2(R_3) \cap \mathscr{L}^2(R_3)$, which implies $u \in \mathscr{D}_A$ and completes the theorem. Theorem 16 is true for $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$ in $\mathscr{D}_{\hat{A}}$ and for A_{FP} and A_{RA} in $\mathscr{D}_A$.

Since the continuum spectrum of A_{FP} in $\mathscr{D}_A$ (and $\bar{A}_{\mathrm{FP}}$ in $\bar{\mathscr{D}}_A$) is empty, Theorem 16 completes the story for the operator A_{FP}. In fact, we even know all the eigenfunctions of A_{FP}. On the other hand, we do not even know whether A_{RA} in $\mathscr{D}_A$ has any eigenfunctions besides those corresponding to the zero eigenvalue. For this reason, it is useful to relate the spectrum of A_{RA} in $\mathscr{D}_A$ to the more general properties of the spectrum of $\bar{A}_{\mathrm{RA}}$ in $\bar{\mathscr{D}}_A$.

It is useful to state the *spectral theorem*[12] for self-adjoint operators at this point.

Theorem 17. Let A in $\mathscr{D}_A$ be self-adjoint. Then, there exists a family of projection operators E_λ in $\mathscr{H}$, $-\infty < \lambda < \infty$, with the following properties:

(1) E_λ is symmetric and $E_\lambda E_\lambda = E_\lambda$.

(2) $E_\lambda E_\mu = E_\mu E_\lambda = E_\sigma$, $\sigma = \min\{\lambda, \mu\}$.

(3) $E_{\mu+0} = E_\mu$ for $-\infty < \mu < \infty$. Here, $E_{\mu+0}$ is defined by $E_{\mu+0}u = \lim_{\lambda\to\mu} E_\lambda u$ with $\lambda \geqslant \mu$.

(4) $\lim_{\lambda\to-\infty} E_\lambda = 0$ and $\lim_{\lambda\to\infty} E_\lambda = E$.

(5) $Au = \int_{-\infty}^{\infty} \lambda \, dE_\lambda u$ for all $u \in \mathscr{D}_A$.

(6) E_λ is uniquely determined by the properties mentioned here.

(7) $(Au, v) = \int_{-\infty}^{\infty} \lambda \, d(E_\lambda u, v)$ for all $u \in \mathscr{D}_A$ and $v \in \mathscr{H}$.

(8) $u \in \mathscr{D}_A$ if and only if the Stieltjes integral $\int_{-\infty}^{\infty} \lambda^2 \, d\rho(\lambda)$ exists, where $\rho(\lambda) = (E_\lambda u, u)$.

(9) $(E_\lambda - E_\mu) u \in \mathscr{D}_A$ for every $u \in \mathscr{H}$ and every finite λ, μ; and $E_\lambda u \in \mathscr{D}_A$ for every $u \in \mathscr{D}_A$ and every λ, $-\infty < \lambda < \infty$.

By the following generalization of Theorem 15, we shall be able to prove that the spectrum of A in $\mathscr{D}_A$ coincides with the spectrum of $\bar{A}$ in $\bar{\mathscr{D}}_A$ when A is of the form of A_{FP} or A_{RA}. The generalization represents a simple extension of that given by Hellwig[12] for the second-order differential operators he treated.

Theorem 18. Let c and λ_0 be fixed real numbers, let $\tilde{G} = \{(x, \lambda) \mid x \in G, -\infty < \lambda < \infty\}$, and let $w_\lambda(\mathbf{x})$ be a function which is locally integrable in $\tilde{G}$, as well as, for fixed λ, in G. Let $a(\mathbf{x})$ and K satisfy the hypotheses of Theorem 15. If, for all $\phi \in \mathscr{C}^\circ(G)$ and every real number in λ

$$\int_G \{-w_\lambda(\mathbf{x}) \, \Delta_{\mathbf{x}} \phi + w_\lambda(\mathbf{x}) \, a(\mathbf{x}) \, \phi(\mathbf{x}) - \phi(\mathbf{x}) \, K[w_\lambda] - \lambda w_\lambda(\mathbf{x}) \, \phi(\mathbf{x})\} \, d\mathbf{x} = \int_G \eta_\lambda(\mathbf{x}) \, \phi(\mathbf{x}) \, d\mathbf{x} \tag{97}$$

where

$$\eta_\lambda(\mathbf{x}) = c \int_{\lambda_0}^{\lambda} w_\mu(\mathbf{x}) \, d\mu$$

then, for every λ, $w_\lambda(\mathbf{x}) = \tilde{w}_\lambda(\mathbf{x})$ almost everywhere in G, where $\tilde{w}_\lambda(\mathbf{x}) \in C^2(G)$.

The details of the proof of this theorem, again with the modifications given in the proof of Theorem 15, follow the details given by Hellwig[12] for the case in which $K \equiv 0$. Thus, we shall omit the proof. Let us present one more definition before using Theorem 18.

In physics, one often uses the concept of eigenpackets to deal with the continuous spectrum of an operator. Following Hellwig's adaption of Hellinger's definition[20] of eigenpackets, we say that Φ_λ *is an eigenpacket of the symmetric operator* A in $\mathscr{D}_A$ if

(1) $\Phi_\lambda \in \mathscr{D}_A$ for $-\infty < \lambda < \infty$, $\Phi_{\lambda_0} = 0$, where λ_0 is some real, fixed number; $\lambda_0 = -\infty$ is admissible;

(2) Φ_λ is continuous in λ, i.e., $\lim_{\lambda \to \mu} \| \Phi_\mu - \Phi_\lambda \| = 0$ for every λ;

(3) $$A\Phi_\lambda = \int_{\lambda_0}^{\lambda} \mu \, d\Phi_\mu \qquad \text{for} \quad -\infty < \lambda < \infty \tag{98}$$

We shall now state some differentiability properties of Φ_λ and the spectral family E_λ.

Theorem 19. If A in $\mathscr{D}_A$ is an operator obeying the hypotheses of Theorem 15, then, for every eigenpacket Φ_λ of $\bar{A}$ in $\bar{\mathscr{D}}_A$,

$$\Phi_\lambda(\mathbf{x}) \in \mathscr{D}_A, \qquad \text{i.e.,} \quad \bar{A}\Phi_\lambda = A\Phi_\lambda \tag{99}$$

Theorem 20. Let A in $\mathscr{D}_A$ be an operator obeying the hypotheses of Theorem 15. Let E_λ in $\mathscr{H}$ be the spectral family associated with $\bar{A}$ in $\bar{\mathscr{D}}_A$. Then, for all $v \in \mathscr{H}$ and all real numbers λ_0, λ,

$$(E_\lambda - E_{\lambda_0})\, v \in \mathscr{D}_A \qquad \text{and} \qquad \int_{\lambda_0}^{\lambda} \mu \, dE_\mu v \in \mathscr{D}_A \tag{100}$$

If A in $\mathscr{D}_A$ is bounded from below, $E_\lambda v \in \mathscr{D}_A$.

These theorems are true for $\hat{A}_{\mathrm{FP}}$ and $\hat{A}_{\mathrm{RA}}$ in $\mathscr{D}_{\hat{A}}$ and for A_{FP} and A_{RP} in $\mathscr{D}_A$.

Aside from using Theorem 18 instead of the similar theorem for $K \equiv 0$, the proofs of these theorems are identical to the proofs of similar theorems given by Hellwig.[12] Thus, we shall omit the proofs.

Finally, we shall establish the theorem which will enable us to prove Theorem 2 of the section on existence theorems.

Theorem 21. Let A in $\mathscr{D}_A$ be an operator obeying the hypotheses of Theorem 15. Let E_λ in $\mathscr{H}$ be the spectral family associated with $\bar{A}$ in $\bar{\mathscr{D}}_A$. Assume, moreover, that $(u, Au) \geqslant b \,\| u \|^2$, $b > 0$, for all $u \in \mathscr{D}_A$. Then, for any fixed numbers $\lambda \geqslant \lambda_0 \geqslant b$, and for any $v \in \mathscr{H}$,

$$\int_{\lambda_0}^{\lambda} (1/\mu) \, dE_\mu v \in \mathscr{D}_A \tag{101}$$

Proof. We need only establish that $\int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v \in C^2(R_n)$, since, clearly, $\int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v \in \mathscr{H}$. Consider $\phi \in \mathscr{C}^\circ(R_n)$. Since $\bar{A}$ is self-adjoint and since

$$\int_{\lambda_0}^{\lambda} (1/\mu) \, dE_\mu v \in \bar{\mathscr{D}}_A ,$$

we may write

$$\begin{aligned} \left(A\phi, \int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v\right) &= \left(\phi, \bar{A} \int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v\right) \\ &= \left(\phi, \int_{\lambda_0}^{\lambda} dE_\mu v\right) \\ &= (\phi, (E_\lambda - E_{\lambda_0})v) \qquad \text{for all} \quad \phi \in \mathscr{C}^\circ(R_n) \end{aligned} \tag{102}$$

According to Theorem 20, $(E_\lambda - E_{\lambda_0})\, v \in C^2(R_n)$, and, consequently, Theorem 15 may be applied to obtain

$$\int_{\lambda_0}^{\lambda} (1/\mu) \, dE_\mu v \in \mathscr{D}_A \tag{103}$$

Moreover,

$$A \int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v = \bar{A} \int_{\lambda_0}^{\lambda} (1/\mu)\, dE_\mu v = \int_{\lambda_0}^{\lambda} dE_\mu v \tag{104}$$

The right most member of Eq. (104) is equal to $E_\lambda v$ if $\lambda_0 = b$ since b represents the lower bound to the spectrum of the A in $\mathscr{D}_A$ hypothesized here.

We can now prove Theorem 2. The operators A_{FP} and A_{RA} in $\mathscr{D}_A$ can be written in the form

$$A = B + C \tag{105}$$

where B is a positive-definite, essentially self-adjoint operator and C is a completely continuous self-adjoint operator. In particular, according to Theorem 10, we can choose

$$B_{\mathrm{FP}} = A_{\mathrm{FP}} + K_{00}\,, \qquad C_{\mathrm{FP}} = -K_{00} \tag{106}$$

and

$$B_{\mathrm{RA}} = A_{\mathrm{FP}} + a\nu(\xi)E, \qquad C_{\mathrm{RA}} = K_1 - K_2 \tag{107}$$

where K_{00} is defined in Theorem 10 and K_1 and K_2 are defined by Eqs. (11) and (12). Consider the equation (note that $\bar{C} = C$)

$$\bar{B}w + Cw = f \tag{108}$$

for $f \in \mathscr{L}^2(R_3\,;\,\omega)$ such that

$$(f, \psi) = 0$$

for any solution to the equation

$$(\bar{B} + C)\,\psi = (B + C)\,\psi = 0 \tag{109}$$

The second equality follows from Theorem 16. According to Theorem 11, a solution $w \in \mathscr{L}^2(R_3\,;\,\omega)$ to Eq. (108) exists. Moreover, according to the spectral theorem

$$w(\xi_1) = -\int_b^\infty (1/\mu)\, dE_\mu Cw + \int_b^\infty (1/\mu)\, dE_\mu f \tag{110}$$

where b is the lower bound on B and E_μ is the spectral family associated with B. Consider next the sequence $\{w_n(\xi_1)\}$, $n = 1, 2, \ldots$,

$$w_n(\xi_1) = -\int_b^n (1/\mu)\, dE_\mu Cw + \int_b^n (1/\mu)\, dE_\mu f \tag{111}$$

According to Theorem 21, $w_n(\xi_1) \in \mathscr{D}_B = \mathscr{D}_A$. Hence,

$$\begin{aligned} Aw_n &= \bar{A}w_n \\ &= (\bar{B} + C)\,w_n \\ &= -\int_b^n dE_\mu Cw + \int_b^n dE_\mu f + Cw_n \end{aligned} \tag{112}$$

$\{w_n(\xi_1)\}$ is a converging sequence,

$$\lim_{n\to\infty} w_n(\xi_1) = w(\xi_1) \tag{113}$$

and, because $\lim_{n\to\infty} Cw_n = C(\lim w_n) = Cw$ for completely continuous operators,

$$\lim_{n\to\infty} Aw_n - f = 0 \tag{114}$$

almost everywhere in R_3. This completes the proof of Theorem 2.

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REFERENCES

1. J. G. Kirkwood, *J. Chem. Phys.* **14**:180 (1946).
2. S. A. Rice and A. R. Allnatt, *J. Chem. Phys.* **34**:2144 (1961).
3. S. A. Rice and P. Gray, *The Statistical Mechanics of Simple Liquids* (John Wiley and Sons, New York, 1965).
4. C. C. Wei and H. T. Davis, *J. Chem. Phys.* **45**:2533 (1966).
5. S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, England, 1939).
6. E. Hecke, *Math. Z.* **12**:274 (1922).
7. T. Carleman, *Problèmes Mathématiques dans la Théorie Cinétique des Gaz* (Almqvist and Wiksells, Uppsala, 1957).
8. H. Grad, *Advan. Appl. Mechanics* **1** (suppl. 2):26 (1963).
9. R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, New York, 1953), Vol. 1.
10. M. O. Baleiko and H. T. Davis, *J. Chem. Phys.* **52**:2427 (1970).
11. B. Friedman, *Principles and Techniques of Applied Mathematics* (John Wiley and Sons, New York, 1956).
12. G. Hellwig, *Differential Operators of Mathematical Physics* (Addison-Wesley, Reading, Mass., 1967).
13. J. L. Lebowitz, H. L. Frisch, and E. Helfand, *Phys. Fluids* **3**:325 (1960).
14. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Frederick Ungar, New York, 1955).
15. G. Hellwig, *Partielle Differentialgleichungen* (Stuttgart, 1960). English translation: *Partial Differential Equations* (Blaisdell, New York, 1964).
16. A. E. Taylor, *Advanced Calculus* (Ginn and Company, Boston, 1955).
17. H. Weyl, *Duke Math. J.* **7**:411 (1940).
18. G. Fichera, *Convegno internazionale sulle equazioni lineari alle derivate parziali* (Trieste, 1954, Rome, 1955).
19. E. Wienholtz, in Ref. 15, Chapter 4, Section 4.
20. E. Hellinger, *J. Reine Angew. Math.* **136**:210 (1909).